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EXAMINATION OF PRE-SERVICE MATHEMATICS TEACHERS' MENTAL CONSTRUCTIONS OF FUNCTION TRANSFORMATION

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Abstract: Learning and teaching function transformation has a significant place since it provides new opportunities for learners to use, reflect and discover knowledge related to the concept of function. This study's main goals related to the importance of learning function transformation are to examine pre-service mathematics teachers' (PMTs) mental constructions of function transformation from APOS theory perspective and identify their concept images and concept definitions of function transformations. Two frameworks, APOS theory and concept images and concept definitions, were considered in this study. The qualitative study adopted case study design. The participants were three female PMTs. Clinical interview methodology was used to collect the main data. Data were analyzed based on the APOS framework, and the descriptive analysis method was used in the process. The results of the study present PMTs' mental constructions as well as to their concept images or definitions of function transformation. In this study it is found that PMTs had limited knowledge regarding function transformation, but their mental constructions were changed throughout the process and they partially overcame their difficulties related to the families of functions and related function concept images and definitions.

Key words: APOS, Concept image and Concept definition, Function transformation, Preservice mathematics teachers

1. Introduction

Teaching of functions and related concepts are core points of mathematics education (National Council of Teachers of Mathematics [NCTM], 2000; Selden & Selden, 1992). Developing a sense for functions is a primary goal of secondary education (Common Core State Standards Initiative [CCSSI], 2010). Function concept also has an integrative role between modern mathematics' topics (Leinhardt, Zaslavsky, & Stein, 1990) and the fundamentals of science (Selden & Selden, 1992). Learning and teaching function transformation is crucial since it provides new opportunities to use, reflect and discover knowledge about the concept of function (Lage & Trigueros, 2006). Function transformation is taught at different grades (Bingham, 2007; Zazkis, Liljedahl, & Gadowsky, 2003). Many countries (e.g., Australia, England, Turkey) give great importance to function transformation in curriculum and standards (Common Core State Standards for Mathematics (CCSSM), 2010; NCTM, 2000). In addition to standards and teaching programs, some researchers examined function concept learning and teaching from a broader perspective (Lage & Trigueros, 2006; Baker, Hemenway, & Trigueros, 2001; Confrey & Smith, 1991; Cooney, Beckmann, Lloyd, Wilson, & Zbiek, 2010). According to Cooney et al. (2010), combining and transforming functions is important for learning function transformation amongst other big ideas including function concept, covariation and rate of change, families of functions, and multiple representation of functions. This idea depends on the properties of addition, subtraction, multiplication, division, taking inverse and composition of other functions. When these computations are applied, the graph of functions can be transformed into another or appear as shrinking or stretching, horizontally or vertically. This complicated big idea is left to last since it requires general understanding of functions combining the other big ideas (Lage & Trigueros, 2006; Baker, Hemenway, & Trigueros, 2001; Confrey & Smith, 1991).

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This study mainly aims to examine pre-service mathematics teachers' (PMTs) mental constructions regarding function transformation from APOS theory perspective, and identify their concept images and definitions of function transformations. In the direction of this aim, the questions, "How do PMTs build their mental constructions of function transformation?" and "what are PMTs' concept images and concept definitions of function transformation?" guided this study. Two theoretical frameworks were used for this study. In previous studies, Dubinsky and colleagues' APOS theory and Tall and Vinner's (1981) concept images and definitions were used separately to analyze learning and teaching of function and related concepts. Also, different frameworks were used with APOS theory to examine function and related concepts. Trigueros and Martínez-Planell (2010) used Duval's semiotic representation theory with APOS theory to examine geometric representation's importance in learning two-variable functions. However, no study has used APOS theory and theory for concept image and definition together for the concept of function. This current study uses these two theoretical frameworks while analyzing PMT's responses, thoughts and strategies on given tasks.

2. Theoretical Framework

2.1. APOS Theory

APOS, which is an acronym for Action (A), Process (P), Object (O), and Schema (S), theory is well-accepted for detailed descriptions of learning many mathematical concepts, especially in secondary education. In APOS theory, Dubinsky and colleagues (Arnon et al., 2014; Asiala et al., 1996; Clark, et al., 1997; Dubinsky, 1991; Weller et al., 2003) extended Piaget's theory of reflective abstraction, applying it to advanced mathematical thinking. Main goal was to create a model to investigate/analyze college students' mental constructions of mathematical concepts.

Action is the lowest level of abstraction, and a significant beginning in learning new concepts. Students use existing knowledge of physical or mental objects in attempting to learn new actions. At this level, external cues detail the steps in performing an operation when students carry out transformations while learning a new concept. Action level students construct transformed functions step-by-step using variables. Previous mental constructions or experiences help choosing next steps in performing an operation. When actions are repeated or reflected, students move from relying on external cues to internal control. Process level is under the students' control, and occurs without external stimulus (Asiala et al., 1996). Process level students can reflect on, describe, or reverse transformation steps without performing them explicitly. Students can mentally shift easily between Action and Process levels once reaching the Process level; skipping or reversing steps through interiorization mechanism (Arnon et al., 2014). For instance, for dynamic transformation, Dubinsky and Harel (1992) indicated that students produced the same transformed quantity based on the original quantity repeated/reflected. Research indicates that students should at least reach Process level to understand the concept of function.

New processes can be encapsulated and transformed into new objects, and newly-constructed objects reverted back to processes. When students see the process as an entity, they recognize that transformations could be acted on it. Then the process is encapsulated into a cognitive object (Asiala et al., 1996). Arnon et al. (2014) defined the mechanism of encapsulation as, "when an individual applies an Action to a process, that is, sees a dynamic structure as a static structure to which actions can be applied" (p.21). Students reaching Object level conception can synthesize related Actions, Processes, and mental objects to form a Schema. For instance, students can use objects (function) and think about function transformation as a complete activity, and act on these objects to construct new objects (new functions) (Dubinsky & Harel, 1992).

Students then organize actions, processes, objects, and prior schemas, into a new schema that accurately accommodates newly-discovered knowledge from the mathematical problem (Clark et al., 1997; Dubinsky, 1991). A schema is a collection of constructed actions, processes, objects and other schemas organized in structured form (Arnon et al., 2014; Asiala et al., 1996, 1997; Dubinsky, 1991). In other words, a schema is the totality of knowledge logically connected to a particular concept and the exact concept dictates the necessity for constructing the schema.

2.2 Concept Image and Concept Definition

Concept images and definitions are crucial to developing students' mental construction of mathematical concepts. Tall and Vinner (1981) stated that students learn mathematical concepts using their individual experiences before seeing the formal concept definition. During concept construction, students choose many mental images from their cognitive structure, and throughout this process they could capture the formal concept definition. This mechanism explains the theory describing concept images and concept definitions. Concept image is defined as "describing the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p.152) and concept definition "a form of words which are used to specify the concept" (Tall & Vinner, 1981, p.153). Tall and Vinner (1981) indicated that personal concept definitions could differ from formal definitions based on personal reconstruction. While mentally constructing mathematical concepts, students choose concept images, then mental pictures related to the concept by matching images with the concept name.

Using concept images or definitions are thought to have a significant place in reaching a particular level in APOS theory. Using APOS theory and concept image/definition theories enables the review of phenomena from two different but complementary perspectives. How PMTs build their understanding about function transformations in APOS theory have already been described and classified (Eisenberg & Dreyfus, 1994; Lage & Trigueros, 2006; Wallace-Gomez, 2014; Weber, 2002).

3. Related Literature

Existing literature separately uses these frameworks to analyze students' or teachers' learning or teaching of function and related concepts. Examining 10th-11th grade students' concept definition and concept images about function, Vinner (1983) reported 14% of 146 students assumed a function as simply an equation, and Thompson (1994) described it as "two written expressions separated by an equals sign" (p.5), known as a student's concept image of a function (Vinner, 1983; Vinner, 1992; Vinner & Dreyfus, 1989). Vinner (1983) identified four main definition categories about function: Dirichlet-Bourbaki, rule of correspondence, algebraic term and formula definition, and definition as mental pictures. Ideally, a student's concept image of a function should align with how they define a function, although not often the case. Vinner and Dreyfus (1989) also reported 57 first-year college students defined functions with the standard correspondence definition, but 56% did not actually use the definition when answering questions; instead deferring to concept images which differed to the original definition. Although many studies exist regarding concept definition and concept images of functions (Dubinsky, 2013; Viirman, Attorps, & Tossavainen, 2010; Vinner & Dreyfus, 1989), neither have been examined focusing on function transformation before.

Functions and function transformation are mostly investigated based on APOS theory (Kabael, 2011; Maharaj, 2010; Martínez-Planell & Cruz Delgado, 2016; Trigueros & Martínez-Planell, 2010; Weber, 2002). Carlson (1998), Nyikahadzoyi (2006) and Reed (2007) identified that even successful university students remained at the Action level. Dubinsky and Wilson (2013) examined high school students' understanding and learning of function concept from the APOS theory perspective, and showed students could improve their knowledge and APOS levels on functions through seven weeks of instructional treatment despite having low level understanding. Dubinsky and Wilson (2013) summarized that most students shared evidence of Process level of function concept.

While many studies (Lage & Trigueros, 2006; Wallace-Gomez, 2014) relate to function transformations regarding $f(x) = x^2$ and $f(x) = |x|$, few studies consider other areas such as logarithmic and exponential functions (Zazkis et al., 2003; Baker et al., 2001). For instance, Weber (2002) examined how students learn exponential and logarithmic functions using APOS theory, and suggested a new theory explaining how students learn these concepts, and a framework that could increase student APOS levels. Existing studies indicated that while students easily understand and act on transformations of linear or quadratic functions, they have difficulty acting on or reasoning other function families (Baker et al., 2001; Eisenberg & Dreyfus, 1994). This difficulty relates to students' APOS levels on function families; while reaching Object level for linear and quadratic functions, they only shared evidence of

Action or Process level for other function types. Therefore, it could be inferred that students had neither conceptualization of transformation, nor saw transformations as "a sequence of two static states rather than as a dynamic process" (Eisenberg & Dreyfus, 1994, p.59).

Analyzing existing literature showed that students experience problems and difficulties in algebraically or geometrically examining function transformations, and learning function transformations accepted as difficult concepts for students (Eisenberg & Dreyfus, 1994). Most studies related to function transformations compared vertical and horizontal, and reported students more successful on vertical rather than horizontal transformations (Borba & Confrey, 1996; Eisenberg & Dreyfus, 1994; Zazkis et al., 2003). Eisenberg and Dreyfus (1994) explored learners' understanding of function transformations, focusing on visualization. They acknowledged difficulties in visualizing horizontal transformation compared to vertical, suggesting "there is much more involved in visually processing the transformation of f(x) to f(x+k) than in visually processing the transformation of f(x) to f(x)+k" (Eisenberg & Dreyfus, 1994, p.8).

Lage and Trigueros (2006) analyzed college students' APOS levels of function transformations through solving tasks related to transformation. Results showed that most students experienced difficulties, with the main reason related to interiorizing the Process level or in encapsulating Process level into Object level. While Lage and Trigueros (2006) identified that students who remained at the Process level tried to understand the values of transformed functions by using tables, they experienced more difficulties in horizontally transforming functions than vertically. Additionally, students mostly used rules and algorithms while solving transformation tasks, and while able to correctly solve the original function family, they could not transform one function into another. Accordingly, Lage and Trigueros (2006) shared the common issues students experienced with transformation as: (1) Identifying properties of given transformations; (2) Seeing which kinds of transformations applied to basic functions; and (3) Predicting how transformations would change certain properties of a given function.

Although existing studies reveal students' learning and APOS levels of function transformation, no research has focused on students' concept images and concept definitions of function transformation. Moreover, no study has used both APOS theory and concept images and definitions to illustrate students' mental constructions of function transformation. The current study aims to contribute to the literature with suggestions for improving students' learning of function transformations.

4. Methodology

This study is designed as a case study employing qualitative research methodology. Case studies examine "in depth a program, event, activity, process or one or more individuals" (Creswell, 2009, p.13), and this study aims to gather deep information on students' cognitive structure about function transformation.

4.1. Participants and Settings

The study was conducted with three pre-service middle school mathematics teachers at a public university in the west cost of Turkey. The participants were selected through convenience sampling from 45 3rd grade PMTs on voluntary basis, since identifying their cognitive structure might help in understanding future teachers' perspectives about mathematics, and it might benefit the researcher's college mathematics course. The participants are all female, and high achievers with AA/BA grades from year-one Calculus course.

Three PMTs were interviewed separately by one of the researchers. Clinical semi-structured interviews were conducted in one-to-one sessions in the researcher's office for 120-150 minutes. One-on-one interviews were used because it was anticipated that they would provide us with more reliable data than small group interviewing. During the interview sessions, the researcher sat with PMT at a table. Each session was video-recorded. A camera was focused on PMT, and it was zoomed in on their responses in order to record the PMT's responses more closely. At times, the recording was stopped to give the interviewee time to rest or think. Each interview contained four main tasks –plus subtasks– involving questions constructed by two researchers based on previous studies and their own teaching experiences.

4.2. Data Collection

The data was gathered by clinical interview methodology (Clements, 2000; Ginsburg, 1981). This study used clinical interviews to gather evidence of PMTs' ways of reasoning/thinking and their level of mental constructions (Clements, 2000). Clinical interview methodology helped the researchers gather evidence of PMTs' thinking and mental processes at the level of authentic ideas and meanings. Questioning exposed hidden structures and processes in their thoughts and ideas as the interviews progressed (Clements, 2000).

Researchers constructed the tasks for data collection, then a pilot study was conducted with a pre-service teacher. According to the pilot interview results, the researchers redesigned the way of posing questions, and the final tasks were constructed with the consensus of two researchers.

Whole interview section constituted the data of the study. The worksheets that include the responses of three PMTs' to the tasks and interviewer field notes were also the data sources. In addition, after each session, the interviewer audio-recorded her own reflections on the interview. The tasks were particularly adapted or designed in order to explore PMTs' mental constructions regarding function transformation (See Appendix). The first task was prepared to reveal PMTs' Action level and concept image/definition of function and function transformation. For example, if PMTs use external cues and some memorized rules to conduct transformation for the first question, it is deduced that s/he is on Action level. In other words, the key construction of that level is using memorized rules without giving meaningful explanation to transform the function.

The second task was designed as two parts. In the first part, PMTs were asked to draw parent functions of quadratic, absolute value, exponential, and root functions, and then to draw vertical/horizontal shifting and vertical/horizontal stretching/shrinking forms of each functions. In the second part, the transformed functions' formulae were given to PMTs without drawings, and they were asked to explain each transformation of each function. These tasks aimed to reveal PMTs' Process level of function transformation. For example, if they present a transition on using external cues to control of actions, they could show behavior consistent with having internalized the meaning of transformation by reconstructing the meanings of transformation, which means they are at the process level. In this level, they could perform and observe shift from parent function to transformed one and vice versa.

First part of the third task, which was developed by the researchers, asked PMTs to identify and write the formula based on the parent functions using transformed function graphs. The second part, which was adapted from Smith's (2009), asked participants to explain the function transformation of the graphs. These questions were designed to identify PMTs' Object level regarding function transformation. These questions led the PMTs encapsulate dynamic structures of transformation which evolved from static form of transformed function's graph. In this level, they could identify types of transformation from the graph and produce the formula of them. Transformation is now could be internalized when it is encapsulated from action level to process level so that the relationship between graphs and formulae could be identified and explained.

The fourth task was designed by the researchers to reveal PMTs' Schema level of the concept. In this level, PMTs were expected to provide a generalization about function transformation for all function families related to what they had done so far. Additionally, they were expected to identify the inverse of each function, explain relationships between transformed function and transformed inverse function, and provide a generalization about inverse function transformation. The researchers aimed to examine continuity of transformation concept though the inverse functions by using these questions. In this level, they are able to encapsulate all levels as a whole by conducting relationships between functions and inverse functions.

4.3. Data Analysis

Data were analyzed based on the APOS framework that utilized scripting, building a table describing evidence points, transcribing video-recorded interview sessions, coding, developing a model for each PMT, identifying common issues, and summarizing the data (Arnon et al., 2014; Asiala et al., 1996). The descriptive analysis method was employed throughout (Yıldırım & Şimşek, 2016).

Initially, interviews were transcribed and analyzed, and then a three-column table was used for each PMT. In Table 1, Column 1 (Interview-transcript) is the transcript event that inferred APOS levels and/or concept images/definitions, Column 2 (Comment) shows descriptions and reasons for interpretations, and Column 3 lists the concept images and concept definitions

Table 1. *Framework for interview data analysis*

Q.2a	Code	Interview-transcript	Comment	Concept Image & Concept Definition
(5) $f(x) = \|x\|$ $g(x) = \|x+1\| - 4$	Process	First I shift this function left one unit. Later, it says I need to shift the function four units down the y-axis.	Burcu presented evidence of Process level. Right after she saw the question, and without drawing a graph as a reference, she internalized to shift the graph of the function on the y-axis; which means vertically by observing numbers of absolute value function and shifting on the x-axis; that is horizontally by observing inside the absolute value function. She may conduct this process by repeating similar procedures, yet she internalized after a while so she could manage same situations.	She considered $y = \|x\|$ as the parent function and easily drew vertical transformation. Graph of $y = \|x\|$ was taken as the concept image for this function family's transformation. Operations near absolute value are signs for shifting on x-axis (horizontally), operations outside of absolute value are signs for shifting on y-axis (vertically), which could be considered as concept definition of function transformation.

Being the major entity in qualitative research, each function family member is considered a unit of analysis in this study. Each PMTs' perceptions and capabilities was discussed for quadratic function, absolute value function, exponential function, root function and logarithmic function separately. APOS theory, concept definition and concept images were considered within the family of functions for each participant.

Codes were assigned based on PMTs' responses. In order to ensure reliability of the data, using Table 1, both researchers coded the transcribed interview and met weekly to discuss each code. Conflicts were discussed, and if the mental constructions identified by the researchers matched, particularly APOS levels and concept images and concept definitions, they moved to the next analysis level; else they revisited the analysis until they reached consensus. Researchers used triangulation to ensure validity of the data. For example, different data sources such as video-recordings, field notes and worksheets were used in the study.

Maxwell's (1992) five criteria were considered for validity. As part of descriptive validity, the events were reported directly since the researchers could watch/repeat recorded interview sessions to manage potential misunderstandings. All descriptions and observations were conducted based on a theory. Regarding to Interpretive validity, the researchers discussed and revised their thoughts on levels of APOS theory until they reached consensus about which level covered the participants' observed mathematical act. Therefore, the applied interpretations were collected through theory-based investigation. For theoretical validity, the current study was constructed on the APOS theory and a well-known framework about identifying concept definition and concept image. These two perspectives helped theorize relationships among transformation concepts within the study's "transformation of function" context. A qualitative study does not concern generalizing results, but the method or structure of the study can still be generalized. The current study' participants and their properties, settings and interventions were explained in close detail so that other researchers could replicate it. As part of evaluative validity, the current study's data analysis and interpretation of the data and results were simultaneously evaluated and interpreted by two researchers, plus one expert helped with consensus on data analysis.

5. Results

This section describes the PMTs' mental constructions and concept images and definitions of function transformations. Each function family was a separate case and each PMT's responses to given tasks are shown with examples in each section.

5.1. Concept Images, Definitions of Function and Function Transformation

This section presents analysis of the responses to definition of function and function transformation. Questions were asked to identify PMTs' concept definitions of function and function transformation as well as concept images if they reveal or draw any figures or images in response to questions. Burcu defined function as, "a correspondence that connects the value of x with the value of y", which was also her concept definition. Burcu's response corresponds with the Drichlet-Bourbaki definition (Vinner, 1989). However, when Burcu was asked about the function transformation, she defined it as shifting a function on the x-y axes. This showed that shifts on axes was her concept definition for function transformation.

When asked for an example of her concept image, Burcu drew a parabola and explained horizontal transformation by conducting an invalid shifting to graph, "*if there is an alteration [adding or subtracting a number] on x-values the movement should be on the x-axis, if there is an alteration on y-values the movement should be on the y-axis*". The function $f(x) = x^2$ was her benchmark concept since she referred back to this function whenever she was asked about transformation. Burcu's strong concept images and definition of quadratic function might affect her APOS level regarding quadratic functions.

Similarly, Zuhal defined function considering domain and range sets which implies correspondence by using Drichlet-Bourbaki definition, but when asked about function transformation, she had no definition. She only gave an example and said, "it is a movement on the axis" and wrote the formula $f(x) = (x-5)^2$ on the worksheet. Different from the other PMTs, Lale explained function as a rule which is "something that connects the value of x the value of y" (Vinner, 1989). She said "I cannot give a definition but I can formulize. I could define as y equals something on x. y-values are changing according to values of x. x can have limitless values, but y-values depend on x". Her response relied more on operational conception of function than structural conception. When asked about function transformation, she tried to explain movement without any deeper explanation. She preferred not to draw graphs to express function transformation but wrote mathematical expressions to define transformation. Lale did not share evidence regarding her concept definition of function transformation, but it can be deduced that an example of function transformation, a kind of mental picture, could be her concept definition according to Tall and Vinner's (1981) definition. Another finding about definition of function transformation is that none of the participants mentioned stretching or shrinking of functions while defining function transformation. All of them concentrated on shifting functions vertically or horizontally.

5.2. Quadratic Function: $f(x)=x^2$

During interviews, in addition to question responses, PMTs drawings were also considered in identifying evidence of concept definitions or concept images. Although all PMTs have a concept image of the quadratic function transformation of this function was problematic since they had no proper concept definition for function transformation.

After Burcu drew $f(x) = x^2$, she was asked to draw the transformed form of f(x), $f(x) = (x-2)^2$. Although she used the former as a reference to define function transformation, she conducted vertical instead of horizontal transformation. When she looked to critique her answer, she preferred to extract the equation and obtain an open quadratic function formula. After that she used discriminant method to find the x-axis intersection points. Although, according to her first interpretation, the graph of the function should have intersecting points on the x-axis, according to her calculation she revealed no x-axis intersection. After drawing the graph using discriminant method, she realized the parent function ($f(x) = x^2$) transformed horizontally not vertically, confusing her reasoning and then drew the function without considering function transformation concepts. For the vertical transformation ($f(x) = (x-2)^2 + 3$), even though she solved her misconception about the transformation, she repeated the same approach and applied discriminant method to identify peak-point of parabola and intersection points. Having completed the graph, although she should have concluded that the graph transformed three units upward on the y-axis, she interpreted the y-intercept as "the graph moves seven units upward on the y-axis". When asked how she achieved this result, she responded, "Since the graph intersected the y-axis at point seven, it should be transformed vertically for seven units on the y-axis". She drew the graph using discriminant method, yet she deduced no generalization for the vertical transformation. Initially, it is thought that she was at the action level regarding function transformation, but throughout the process she repeated her actions which were internalized and reached the process level regarding function transformation In other words, since she used mathematical computation and evaluation to draw graphs, the act could be improved by repeating and reflecting her mental constructions regarding transformation, it is deduced she was in Process level for $f(x) = (x \pm a)^2 \pm b$ form of transformations. Although she had wrong generalization about vertical and horizontal shifting of functions, the interview tasks and questions successfully triggered her explorations. In the following task, she correctly moved the function horizontally or vertically.

However, Lale only shared evidence of Action level for horizontal and vertical transformation of quadratic function since she preferred to draw the graph using point-by-point, which is conceived as externally driven. In other words, she found x and y values of the function to draw the transformed graph. Before applying the point-by-point approach, she deduced the graph should be transformed horizontally on left side of x-axes for $f(x) = (x-2)^2$ because of the minus sign inside the parenthesis. After drawing the graph, she changed her mind to the correct conclusion. She revealed evidence for Action level since she constructed a result and a generalization for vertical transformation by using point-by-point approach.

Zuhal also remained at the Action level since she exposed an external influence, saying "*I am trying to remember this process but I cannot*". Although she was at the Action level for horizontal transformation, for horizontal transformation she found peak-point and intercepts points of parabola to draw the graph (Figure 1). She completed this process step-by-step by relying on a memory, which was the external cue for her.

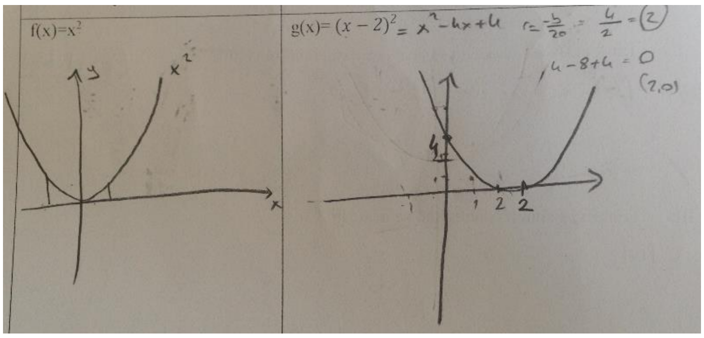

Figure 1. *Zuhal's approach to finding peak-point*

For the graphs of stretching and shrinking transformation types, Burcu revealed evidence for Process level, although she did not apply a correct deduction for a function, multiplied by a constant c, stating "*if the coefficient of x^2 is bigger than 1, then the graph of the function is getting narrower which means stretching*". Since she internalized the rule (if c>1 then the function is horizontally stretching) depending on her previous mathematics courses, she was at the Process level. However, Lale, could not decide how the constant c was affecting the graph of the function. She applied point-by-point approach to draw the transformed function, and therefore it is deduced that she was at Action level. However, even though she provided an evidence regarding action level, Lale exposed an unplanned training throughout the interview and she reached first at Process and then at Object levels of the theory for $f(x) = x^2$. She accepted $f(x) = x^2$ as a reference function and drew with on the given graph to reason about the parent function. She, then, explained which kind of transformation was applied to given functions by examining the formula and driving the formula from the given graphs, even in stretched or shrinking forms. She identified the amount of stretching or shrinking by considering fix points of the graph for both transformed and parent functions (Figure 2).

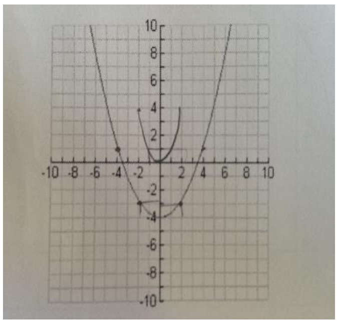

Figure 2. *Lale's work on identifying amount of shrinking*

Zuhal was also at Process level since she realized the graph should be shrink or stretch and examined different x-values for the same y-value both transformed and parent functions. She declared a function multiplied with a constant c did not imply a movement through x or y axes.

5.2. Absolute Value Function: $f(x) = |x|$

When Burcu was asked to draw the absolute value and function, she drew the parent function by considering $y = x$ line and symmetry concept. Drawing the parent function explained her concept image while considering the drawn function and symmetry concept revealed her concept definition of the absolute value function. Burcu was initially confused about the plus sign ($f(x) = |x + 1|$) moving the function right of the x-axis or left. However, after writing as a piecewise function, she realized the plus sign shifted the function left of the x-axis and strategized that the root of the function helped her understand the direction of movement on the x-axis ($x + 1 = 0, x = -1$). Having produced a strategy for this transformation, her APOS theory level was classified as Process. She used the same piecewise function to identify vertical transformation; easily concluding movement on the y-axis. Therefore, her level is again defined as Process level.

Lale was at Action level for this function family and horizontal transformation since she used point-by-point approach to construct the graph and draw her conclusion. For vertical transformation she considered the first graph she drew (top of Figure 3) and dragged points on the graph as the transformation amount down the y-axis. Therefore, it is concluded that she shared evidence for the Process level regarding the transformation.

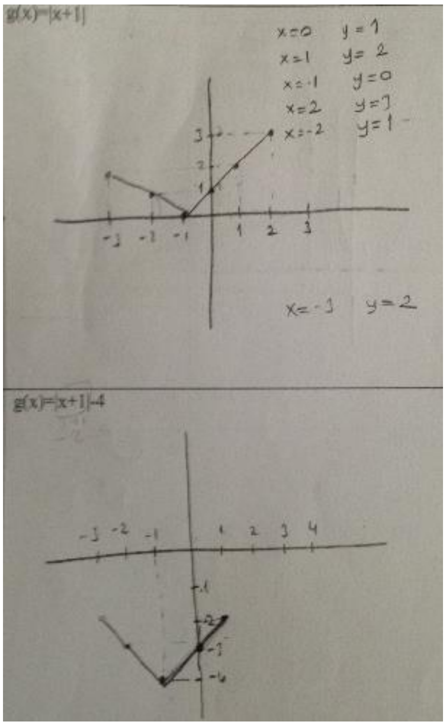

Figure 3. *Lale's transformed absolute value function*

Zuhal's concept definition depended on absolute value, but algebraic not functional notation. However, her concept image on absolute value function evolved on the symmetry of $y = x$. In applying horizontal transformation, she first considered absolute value function as a piecewise defined function and then considered the equation root as zero. Therefore, she first presented Action level properties, but immediately after reached Process level and internalized the relationships of the concepts of root, getting zero and transformation. For vertical transformation ($f(x) = |x + 1| - 4$), she immediately decided that (-4) shifted function down of the y-axis, then checked her answer by considering piecewise defined function. Since she applied mathematical reasoning she acted at Process level.

In this functional family, Zuhal used point-by-point approach to decide which transformation (stretching or shrinking) could be applied to the function according to given formula of the function. Since she did not internalize transformation on vertical/horizontal stretching/shrinking, her decision was affected by external factors; which infers she was at the Action level. Zuhal conducted algebraic computation for some specific points, but could not identify transformation (stretching or shrinking).

The coefficient was $\frac{1}{2}$ (< 1) for function $f(x) = \frac{1}{2}|x + 1| - 4$, so Zuhal concluded the function could shrink. Therefore, according to her reaction, she was classified as Process level. In addition, her concept definition and concept image of absolute value function triggered the vertical and horizontal transformation and she applied some strategies and computations. She found the formula for transformed function graphs and explained each transformation when a transformed function's graph was given. This inferred that Zuhal's mental constructions regarding function transformation evolved in Object level for vertical and horizontal transformation (shrinking/stretching) during the interview. Similarly, Lale presented evidence about progressing from Process level to Object level (Figure 4). She learned and progressed from previous quadratic function and easily identified factor $\frac{1}{2}$ vertically stretches absolute value function.

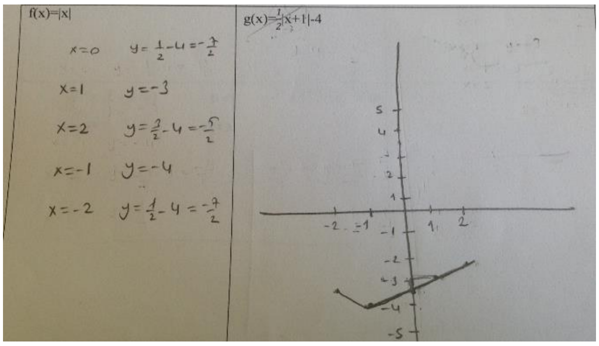

Figure 4. *Lale's absolute value function investigation on stretching*

Lale first stated that the effect of $\frac{1}{2}$ would be to vertically shrink the graph and justified her decision using point-by-point approach, which provided an evidence for her action level. Then, for a graph of absolute value function she decided whether there was stretch or shrink by only observing changes among y-values for the same x-values. In other words, Lale not only identified shrink or stretch, but could also calculate the amount of stretch or shrink, internally.

As a result, all PMTs shared evidence for Object level for absolute value function regarding vertical and horizontal transformation (shifting/shrinking/stretching). They were either at this level from the beginning or reached this level during interview. However, for vertical/horizontal stretching or shrinking, PMTs differed based on their level of understanding. For example, Lale's improvement may depend on her concept definition and concept image of function domain and range since she interpreted y-value changes between parent and transformed function. It is thought that this capability needed strong understanding and concept definition of function and specifically comprehension of a function's domain and range.

5.3. Exponential Function: $f(x) = 10^x$

No participant shared evidence for concept definition or concept image of the exponential function. Burcu drew the parent function based on the graph of $f(x) = e^x$ since she only had a concept image for the natural exponential function, and no concept definition, domain or range for this function. Therefore, she had difficulties drawing the parent function. Similarly, Lale had no concept image about exponential function and used point-by-point approach to reason about the question; although this time it did not work. Even though she tried to draw the graph by giving values to the function, she was not able to draw a graph for exponential function as she did in the previous function transformations. Zuhal also experienced similar difficulties, first drawing the function like a linear function on the first quadrant of the coordinate system, but when questioned about the function's properties she realized the graph was wrong for the second quadrant. Zuhal had no concept image or concept definition for the exponential function.

Since Burcu could not shift function vertically or horizontally, she tried identifying point values and decided for transformations, but then stopped. She did not internalize vertical and horizontal transformation, therefore she remained at the Action level. The same level was observed for Lale; who tried interpreting the function's formula to draw but failed. However, during the interview, for identifying a formula from a graph, Lale recognized vertical transformation and could write and explain the function's formula; therefore, she progressed to Process level. Although Lale progressed, Zuhal did not. She tried to investigate some points on the parent function to draw the transformed function $f(x) = 10^{x-4} - 3$. She determined the graph of transformed function by using point-by-point which was categorized as an external factor for identifying transformation, and therefore was Action level.

For vertical/horizontal shrinking or stretching of the exponential function, Burcu and Lale could not explain or draw the function. However, Zuhal said "*there should be stretching since* $\frac{1}{3}$ *makes the function stretch, I mean I need bigger x-values for the same y-value, therefore function graph is lying on the x-*

axis that is stretching", which means she internalized stretching, but not exponential function. She could not determine an exponential function's formula from a graph, therefore, it is inferred she acted at the Action level for this transformation. The participants performed poorer for this function family than the previous one. This may be due to none of them having a concept image or concept definition of an exponential function.

5.4. Root Function: $f(x) = \sqrt{x}$

Root function was another difficult function for the PMTs as none had a concept definition or concept image of this function. Like Zuhal, Lale preferred to draw the function by using point-by-point approach. Burcu used inverse of root function which was a quadratic function, and since Burcu's concept image and concept definition of quadratic function was better than the other function families, she preferred to use it instead of root function in response to this task. However, Zuhal had no concept image about root function since she used point-by-point approach to draw function $f(x) = \sqrt{x+4}$, but she did not use any negative numbers for x. When asked why, she responded, "in the square root there are no negative numbers". From this explanation, Zuhal understood partial, incomplete information about roots mostly dependent on algebraic reasons unrelated to the function concept. Moreover, she preferred not to draw function $f(x) = \sqrt{x+4}$, on the second quadrant because she claimed there were negative x-values, which root function could not have. These signaled an immature concept image about root function.

The PMTs demonstrated evidence of Process level for vertical and horizontal transformation mostly due to subconsciously being trained during the interviews. Lale internalized the vertical and horizontal transformation and could draw the transformed root function, even explaining the transformations. Similarly, Zuhal also shifted some specific points to explain transformation of whole function. Although Burcu used a quadratic function to explain and draw the root function after the parent function for transformation, she used mathematical reasoning (finding quadratic function peak-point) to express horizontal transformation, which shows her action level. Then, she applied the same reasoning as she did for quadratic function to reason about the function transformation of root function, which provided an evidence for process level. In other words, she internalized transformations about root function to be at Process level. For finding transformed formula of a function, Burcu still used the inverse function relationship between root and quadratic function and performed quite well in switching her reasoning. Therefore she reached Object level for the $f(x) = \sqrt{x \pm a} \pm b$ type of transformation. However, Lale was at Process level; she dragged some important point to identify the new transformed graph and explained vertical/horizontal transformation, internalizing these transformations. This internalization was produced during the interview as she clearly progressed during the session. Although Zuhal claimed vertical transformation for $f(x) = \sqrt{x+4}$, when she found points on the transformed function she realized the transformation was on the x-axis not y-axis. Her point-by-point transformation of one graph to another showed that Zuhal improved her mental construction from Action to Process level.

For stretching and shrinking transformation all PMTs were Action level. Their most important problem was describing stretching or shrinking as "*legs of functions approaches y-axis, like in the parabola*". However, the root function does not have two branches. Since participants' concept image of shrinking or stretching transformation depended on only parabola, they had difficulty expressing other function families' stretching/shrinking movements. For instance, Burcu used quadratic function to express vertical and horizontal transformation, so she was confused about stretching/shrinking transformation. For the parabola, it was easy to express shrinking or stretching transformation because the graph pulled towards x-axis for constant c of $y = cf(x)$ in which $0 < c < 1$. However, the root function was not even and not symmetric on the y-axis. It was very difficult for participants to express pulling or pushing the function towards x or y axes.

Zuhal and Lale stated there should be stretching for function $f(x) = 3\sqrt{x+4} - 2$ because the constant c was greater than 1. However, their conclusion was not due to internalizing root function and stretching, but from learning about condition for the constant c and how c affects function graphs. Therefore, they were both Action level. However, Zuhal gave a remarkable answer to explain why the function shrank; "*I mean if I give 3 for x, the y-value is getting bigger so that I should give a rational number for x which means the function graph is shrinking*". However, she was then confused about shrinking, not giving any axis reference like shrinking horizontally or vertically. She determined the function was shrinking when c was smaller than 1, and therefore could not express it properly.

In this function family, the PMTs presented properties of Action, Process and Object levels. It is thought the low levels were due to lack of concept definition and concept images about root function, and domain and range sets of root functions.

5.5. Logarithmic Function: $f(x) = \log x$

No PMTs had concept definition or concept images about logarithmic function. Moreover, Lale could neither draw the graph of $f(x) = \log x$ nor explain any transformation. Although Zuhal had no concept images of logarithmic function, she used algebraic properties of logarithm and exponent relationship to find some points on the coordinate system and then tried to draw transformed function (Figure 5).

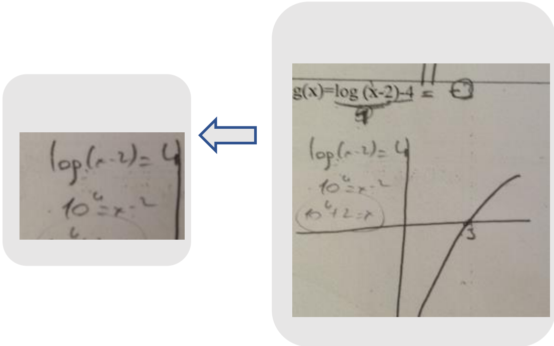

Figure 5. *Zuhal's examination on logarithmic function based on exponents*

Burcu also used the relationship between logarithm and exponents. She knew that inverse function of exponential function was logarithmic function, so first drew the exponential function she previously learned and then produced symmetry about $y = x$ axis and drew a graph of logarithmic function. Although working properly on mathematical properties and reasoning, she did not present any concept image about logarithmic function. She drew horizontally transformed function using point-by-point approach and also exponential relationships. She was Process level since she used switching between exponents to logarithmic for both vertical and horizontal transformation. In other words, she used exponent function transformation to reason about the logarithmic function transformation, which shows that she could reflect her reasoning to another type of function transformation. Zuhal also used this switching option, but she used zero value for y components of point coordinates to identify x-intercepts for the function for drawing the transformed function, which showed she developed point approach during the interview. This implied she was actually at Action level, then progressed to Process level. Similarly, for vertical transformation; she tried to identify the zero values of x to find the corresponding y-value.

For stretching or shrinking, both Zuhal and Burcu remained at Action level. Both employed memorized rules, considered as external factors. Burcu could not draw any graph and stuck with algebraic computation of logarithm and exponents, but Zuhal could only state "*there should be shrinking, but I could not draw the graph*".

For this function family, generally participants remained at Action level. The reason may depend on their lack of concept images and concept definitions. With probing questions and learning during interview sessions, although they drew some graphs and explained the transformation, they lacked confidence in their knowledge and explanations; confessing inadequate knowledge about logarithmic functions for parent function graphs or transformed versions.

5.6. Generalization of Transformation for Function Families

Burcu gave the most competent explanation about inverse function transformation and relationships between function and inverse. She used x-y axes vice versa for the transformation generalization of function, saying "when inside the function differs, the graph should move on x-axis" and "for the inverse function, when inside the function differs, the graph should move on the y-axis".

Burcu already used inverse function relationships in the previous task and here deduced a remarkable connection between transformation of the function itself and transformation of the inverse function. Her concept definition and concept images of inverse function strongly developed. While drawing the logarithmic function, she first drew the exponential function and y=x line in symmetry to the exponential function to get the logarithmic function. She used both algebraic and geometric sense of inverse relationships. For the inverse function transformation, she observed shifting on the x-axis for the function itself implies movement on the y-axis for the inverse function. This deduction with decomposition and recomposition of the concepts, producing her own knowledge and internalization of inverse functions helped her reach Object level.

The others were less successful with inverse function transformations. Lale made no comment about inverse functions; perhaps she had no concept images or concept definition on inverse functions as graphs or algebraic forms. Similarly, Zuhal gave the same explanations with the generalization for functions, saying "all are the same, because x and y are changing for the inverse function and when the graph moves on the x-axis the x-values inside of the function are going to change". She thought that inverse function produced a new function unrelated to the original and transformation of the new function was unrelated to the older original. Therefore, she was at Action level.

Table 2 presents the PMT's APOS levels for function families and transformation types. Where two levels are given, this means the PMT is transiting between levels. For example, Lale demonstrated Object level evidence right after Process level for an absolute value function, hence it is shown as "Process-Object".

Table 2. *Summary of all cases*

Function type	Transformation type	Burcu	Lale	Zuhal
$f(x) = x^2$	Horizontal	Process	Action	Action
	Vertical	Process	Action	Action
	Stretching/Shrinking	Process	Action	Process
	OVERALL	Object	Action	Process
$f(x) = \lvert x \rvert$	Horizontal	Process	Process-Object	Action-process
	Vertical	Process-Object	Process-Object	Action-Process
	Stretching/Shrinking	Process	Process-Object	Action
	OVERALL	Object	Process-Object	Process-Object
$f(x) = 10^x$	Horizontal	Action	Process	Action
	Vertical	Action	Process	Action
	Stretching/Shrinking	-	-	Action
	OVERALL	Action	Process	Action
	Horizontal	Process	Process	Action

Function type	Transformation type	Burcu	Lale	Zuhal
$f(x) = \sqrt{x}$	Vertical	Process-Object	Process	Action-Process
	Stretching/Shrinking	Action	Action	Action
	OVERALL	Object	Process	Action
$f(x) = \log x$	Horizontal	Process	-	Action-Process
	Vertical	Process	-	Action-Process
	Stretching/Shrinking	Action	-	Action
	OVERALL	Action	-	Process
General rule $f(x)$	Vertical/Horizontal transformation	Process	Action-Process	Process
	Stretching or Shrinking	Action	Process-Object	Action
General rule f inverse of x	-	Object	-	Action

6. Discussion

According to this study's results, each participant had different concept definition and concept images for each function family (quadratic, absolute value, exponential, root, and logarithmic functions). Two defined a function as a correspondence that connecting x and y values, also known as the Drichlet-Bourbaki definition (Vinner & Dreyfus, 1989), whilst the other defined function as a rule. The rule-based definition was a category stated by Vinner and Dreyfus (1989) and Viirman et al. (2010). Being one of the most observed definitions in both the literature and textbooks (Akkoç & Tall, 2005; Hatisaru & Erbas, 2017; Vinner & Dreyfus, 1989), using Drichlet-Bourbaki definition more than other definitions was a usual finding. As Viirman et al. (2010) stated, the structural Bourbaki definition was used in upper-secondary schools (Akkoç & Tall, 2005; Hatisaru & Erbas, 2017), but in their study, Viirman et al. (2010) found most of their sample (student teachers and engineering students) used rule-based definition as defined in their textbooks.

In the current study, none of the PMTs could properly explain function transformation. This finding aligns with Kimani's (2008) results in which many 1st-grade calculus students minimally understood function transformations, compositions, inversions, and the relationships between them. Participants' vague explanations of function transformation were mostly based on explanation of "movements on the axis". However, no participants used stretching or shrinking in defining function transformation. This lack of explanation on function transformation may be evolved because of lack of conceptual discussion about it in the mathematics courses. The function transformation is only a transition topic through function definition to graphs of functions. Most of the time function transformation is not the main concept for the mathematics course.

Unfortunately, a concept image of function transformation was only observed with one PMT, and two gave only weak algebraic explanations to describe transformation. Since the lack of conceptual discussion on function transformation in their previous mathematics courses, PMTs did not present any concept images. Rather than giving a general picture of concept of a function transformation, PMTs preferred drawing a parabola as an example in order to describe function transformation. Although one participant had no strong concept images about quadratic function, she strictly followed a formula for finding the peak-point and intercept points of the function while drawing a graph of transformed function. This emphasizes that visualization of transformed functions may depend on algebraic conclusion, which could be a concept definition for the transformation, more than mental pictures. In other words, participant's concept definition of the concept helped her to explain and reason about the function transformation. Many researchers proposed that visualization, representations and conceptual understanding can be considered as related with the process of mathematical learning (Arcavi, 2003; Gutierrez, 1996; Hitt, 2002; Van Nes & De Lange, 2007; Martínez-Planell & Cruz Delgado, 2016; Trigueros & Martínez-Planell, 2010). According to Duval (2006), reasoning and visualization are complementary thought processes. In the current study, not producing concept images of a function meant being unable to continue to explain function transformation. Lale's case with logarithmic function represented Duval's deduction. Although she evolved during the interview and even reached Object level for other function families, she could not implement any transformation for logarithm function since she could not visualize it. Therefore, it could be deduced that concept images might trigger to reach the specific levels regarding APOS theory.

Participants tended to think more algebraically than graphically for function transformation. This might relate with their previous academic life where strong algebraic success was demanded as also discussed by Eisenberg and Dreyfus (1994) and Kaldrimidou and Iconomou (1998) who asserted that secondary-level teachers traditionally fixated their teaching on algebraic representations of functions, rather than from the graphical perspective (Eisenberg & Dreyfus, 1994; Kaldrimidou &Iconomou, 1998). Markovits, Eylon, and Bruckheimer (1986) observed that translation from graphical to algebraic form was more difficult than the reverse conversion, and examples given by learners were limited in graphical and algebraic form. This finding parallels the current study's results about PMTs using algebraic representation for explaining function transformation, instead of graphical representation for defining function transformation. While teaching/learning function transformation using multiple representations (graphical, algebraic and tabular) is appreciated and encouraged (Larson & Hostetler, 2007), none of the current study's PMTs used tabular representation to connect transformed and parent functions or sought help from tabular investigation to procure transformed functions' formula.

When the sessions started, all PMTs drew horizontally transformed functions as if vertical. This might be due to their original academic learning order of transformation. Function transformation was taught in the order of vertical shifting and horizontal shifting in precalculus and secondary school mathematics textbooks (Boz & Erbilgin, 2015; Özkan, 2016), and most students learned this concept through memorization (Zazkis, Liljedahl, & Gadowsky, 2003).

Although PMTs used either discriminant formula or point-by-point approach to check correctness of the transformation they performed, all had some confusion about the minus sign which makes the function move right side of the x-axis. This finding aligns with many other studies (Baker et al., 2001; Borba & Confrey, 1996; Lage &Trigueros, 2006; Zazkis et al., 2003). This confusion was observed since the participants memorized the rules of vertical transformation and applied it to all forms of transformation. Several studies also found that certain transformations proved more difficult to comprehend than others. For example, Lage and Trigueros (2006) observed that students saw visual transformations as something happening to the whole function, having failed to consider how the transformation would change each point on the graph. This led to students experiencing difficulty with "rigid" transformations, or those that moved the graph (e.g., left, right, up or down), but retained the exact same shape. Horizontal and vertical transformation was easier for PMTs than stretching or shrinking. Horizontal transformation was considered the more difficult of the two (Borba & Confrey, 1996; Eisenberg & Dreyfus, 1994; Lage & Trigueros, 2006; Baker et al., 2001); however, "dynamic" transformations, or those that changed the graph's shape (e.g., stretching/shrinking), were deemed much more difficult, as not every point was transformed exactly the same. Also, several studies (Borba & Confrey, 1996; Eisenberg & Dreyfus, 1994; Zazkis et al., 2003) reported students' ease with vertical translations moving the graph in the direction expected (e.g., $f(x)+3$ moves the graph up three units). However, they experienced much more difficulty with horizontal translations moving the graph counterintuitively (e.g., $f(x+3)$ moves the graph three units left, not right). Zazkis et al. (2003) found students who comprehended this concept did so relying on memorized rules (e.g. "plus moves it to the left, minus to the right"), and did not try to explore why this happened, even when prompted.

At the start, PMTs had difficulties with stretching/shrinking transformation types, but they evolved during the interviews. The most obvious development observed was for Lale who, although she has no concept image or concept definition about function transformation, developed a strategy to identify the amount of stretching/shrinking by considering parent and transformed function graphs. This lead a conclusion, both the researcher and the tasks might also affect participant responses that evolved during questioning. Tasks were designed as sequential; from simple to difficult, and each solution requires knowledge of the previous task's solution. In other words, it is thought that task design and interview questions might affect APOS levels as well as concept images and concept definitions during interviews. This gives a clue about improving the comprehension on function transformation that is even a short period of training could develop the understanding on function transformation.

According to results PMTs depended more on algebraic examination of formula in graph drawing, and could not use concept images of the parent function. This finding aligns with Eisenberg and Dreyfus' (1994) study which reveals that while visual methods help students understand transformations, they prefer solving problems algebraically. Possible reasons include students are generally more fluent with algebraic methods, and naturally struggle with graphical methods (Lage & Trigueros, 2006).

Interviews also identified that when students were asked about stretching/shrinking transformed functions, they always used terms regarding functions' arms as opening or closing, even though the function had no arms. The reason could relate to students leaning quadratic functions, since in Turkey, teaching function transformations is in high school or later. Explanations of transformations were insufficient and illogical while analyzing logarithmic and root functions without arms. In these situations, it is perhaps necessary to indicate definitions by referring to x-y axes such as the function approaches to the x- or y-axis. Because of this deficiency, participants could not provide meaningful explanations regarding function transformation of $cf(x)$ by using constant c and analyzing the function algebraically. This situation could be linked to participants' inadequacies in concept images and definitions of functions in general, particularly regarding the stretching/shrinking of transformed functions.

Among the function families, exponential and logarithmic functions were the two most difficult for the PMTs. One preferred not to draw graphs of these two functions, and the other two mostly made deductions depending on algebraic properties of exponent and logarithm. They did not present no concept images for both functions. According to Eisenberg (1992), students did not consider graphical solutions when solving problems, which happens even for problems designed to force them to think this way. The current study also reflected this observation; although tasks were dependent on drawing or reading graphs, the PMTs insisted on making interpretations based on algebraic results.

Another difficulty for PMTs was root function. Although more familiar than exponential and logarithm, the PMTs could not draw its graph. This might relate to their lack of concept images about root function. One PMT used point-by-point approach, but also considered the algebraic properties of root operation not root function or domain of a function. Another PMT used inverse of root function which is quadratic function. According to Baker et al. (2001) and Lage and Trigueros (2006), students better understand transformation when applied functions with which they have more familiarity such as quadratic functions, and had difficulties with simple yet unfamiliar functions such as square root.

Using inverse function was observed for root and logarithmic function for one PMT. Although Carlson, Oehrtman, and Engelke (2010) revealed that finding an inverse function is not an easy concept and in their sample a very low percentage of precalculus students correctly determined the inverse of a function for a specific value, in this study one PMT efficiently used inverse functions to analyze function transformations. However, using inverse function rather than the function itself was evidence of lacking concept images about root and logarithm functions. Although using inverse functions were challenging according to the literature (Breen, Larson, O'Shea, & Pettersson, 2016; Carlson, Oehrtman & Engelke, 2010; Even, 1992), this PMT skillfully used the inverses to produce transformed functions. By doing the switching, she also switched the transformation rules depending on the variables. Even though finding inverse of a function is considered more challenging, students choose finding of an inverse rather than using function transformation. This could be explained by considering the topics of mathematics courses. The inverse function and related topics take a very large part of the course on the other hand function transformation on root and logarithmic function would not be discussed in the mathematics courses, in detail. Therefore, this replacement with function transformation of root and logarithmic function with finding inverses of them and transformed the inverses functions is an expected finding in the context of Turkish mathematics curriculum.

None of the PMTs shared evidence for schema-level function transformation. At this level, participants were expected to construct interconnections among Action, Process and Object levels and reconstruct their own knowledge about function transformation. This is the topmost level for any topic and requires sophisticated knowledge. Baker et al. (2001) suggested special care be taken when teaching function transformation to ensure that students acquire ample sophisticated knowledge of function. Similar suggestions by Eisenberg and Dreyfus (1994) stated that an object conception of function might be a prerequisite to effective understanding of transformations. However, not only object concept of function but also a strong concept image of function transformation is needed for effective function transformation. In general, this current study concludes that preservice teachers should have a strong concept image and concept definition of function transformation which might affect their APOS levels of the concept, plus understanding of function to fully understand the concept of transformation.

7. Suggestions and Implications

This study revealed that comprehension on function transformation could be improved through carefully prepared tasks and detailed discussion. It is suggested that textbook authors and curriculum developers prepare context of function transformation in sequential order considering their topics. The main implication of this study for PMTs is their difficulty with function transformation and concept definitions on families of functions. As future mathematics teachers, they should be more competent about functions and related topics in order to create mathematical productive classroom environments. Special attention should be given to function transformation in teaching, consisting of multiple representations (algebraic, visual and tabular) and axis-based explanations (not using specific functions properties like arms of a parabola), especially for stretching and shrinking transformation. Another implication of this study is that interview process helped PMTs to progress on function transformation so that teachers could give concentrated training on function transformation in a sequential way to develop comprehension on the topics.

The most important direction for future studies would be to use this study's results to produce teaching modules for teaching function transformation to preservice mathematics teachers, considering the importance of APOS theory and concept images and concept definitions.

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Appendix

TASKS

1- Answer the following questions.

a. Define function?

b. What can we do to transform a function to another one?

c. What does transformation mean?

d. How you can transform $f(x) = x^2$function to another one, explain. (apply a transformation)

2- a. In the right column draw the transformed function using the main function.

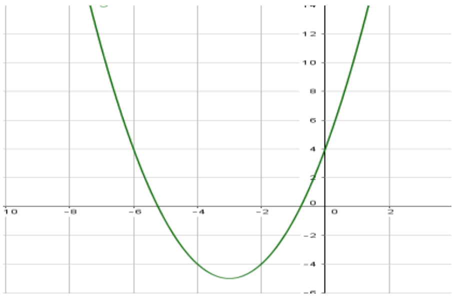
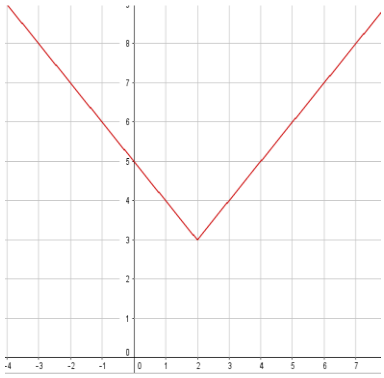
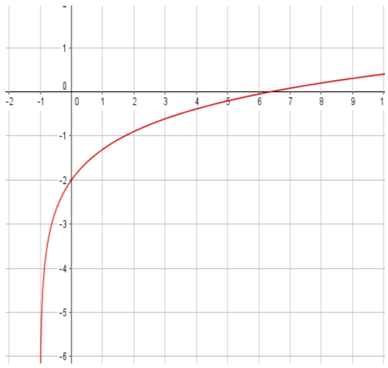
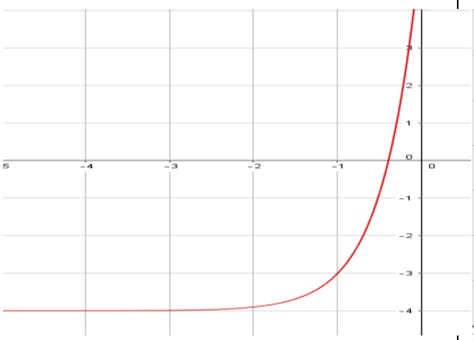
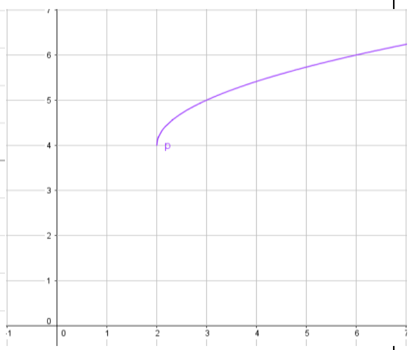
Main Function	Transformed function
$f(x) = x^2$	$g(x) = (x-2)^2$

$f(x) = x^2$	$g(x) = (x-2)^2 + 3$
$f(x) = x^2$	$g(x) = 2(x-2)^2 + 3$
$f(x) = \|x\|$	$g(x) = \|x+1\|$
$f(x) = \|x\|$	$g(x) = \|x+1\| - 4$
$f(x) = \|x\|$	$g(x) = \frac{1}{2}\|x+1\| - 4$
$f(x) = 10^x$	$g(x) = 10^{x-4}$
$f(x) = 10^x$	$g(x) = 10^{x-4} - 3$
$f(x) = 10^x$	$g(x) = \frac{1}{3}(10^{x-4}) - 3$
$f(x) = \sqrt{x}$	$g(x) = \sqrt{x+4}$
$f(x) = \sqrt{x}$	$g(x) = \sqrt{x+4} - 2$
$f(x) = \sqrt{x}$	$g(x) = 3\sqrt{x+4} - 2$
$f(x) = \log x$	$f(x) = \log(x-2)$
$f(x) = \log x$	$f(x) = \log(x-2) - 4$
$f(x) = \log x$	$f(x) = 4\log(x-2) - 4$

b. Specify the main function and explain which transformations are applied to the transformed function.

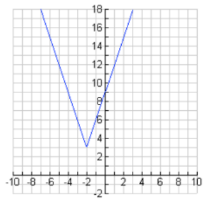
#	Transformed function	Main function	Which transformations are conducted?
1.	$y = \frac{1}{2}x^2$		
2.	$f(x) = \frac{1}{4}(x+3)^2$		
3.	$f(x) = 4\sqrt{x-1} + 2$		
4.	$f(x) = 10^{\frac{x}{4}}$		
5.	$y = \frac{1}{2}(10^{x-1})$		
6.	$f(x) = 3\|x+2\| - 1$		
7.	$f(x) = \frac{1}{2}\sqrt{x+2}$		
8.	$y = \left\|\frac{x}{4}\right\|$		
9.	$f(x) = \log 5x$		
10.	$f(x) = \log(x-1)$		

3- a. Find the formula of the given functions

		
f(x)=	f(x)=	f(x)=
		
f(x)=	f(x)=	

b. From the transformed functions graphs, explain which transformations are conducted to the main function.

1.



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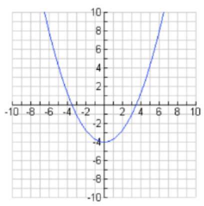

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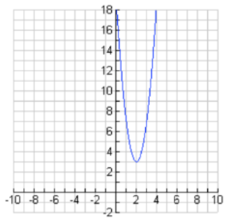

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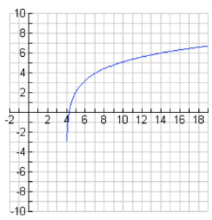

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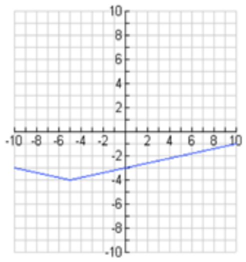

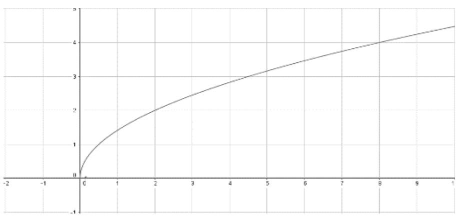

6.

4- Answer the following questions.

i) What is the generalization of the function family: $f(x) = x^2$, $f(x) = |x|$, $f(x) = \sqrt{x}$, $f(x) = \log x$ and $f(x) = 10^x$ transformation?

For each function (ii-vi), find the inverse, then find one of the transformed functions of that inverse function:

ii) $f(x) = x^2$

iii) $f(x) = |x|$

iv) $f(x) = \sqrt{x}$

v) $f(x) = \log x$

vi) $f(x) = 10^x$

vii) What generalization can you produce about these inverse function transformations?

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